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Mean Squared Error Performance Prediction of Maximum-Likelihood Signal Parameter Estimation

Christ D. Richmond

Adaptive Senor Array Processing Workshop
Session III: Adaptive Detection and Estimation

12th March 2003

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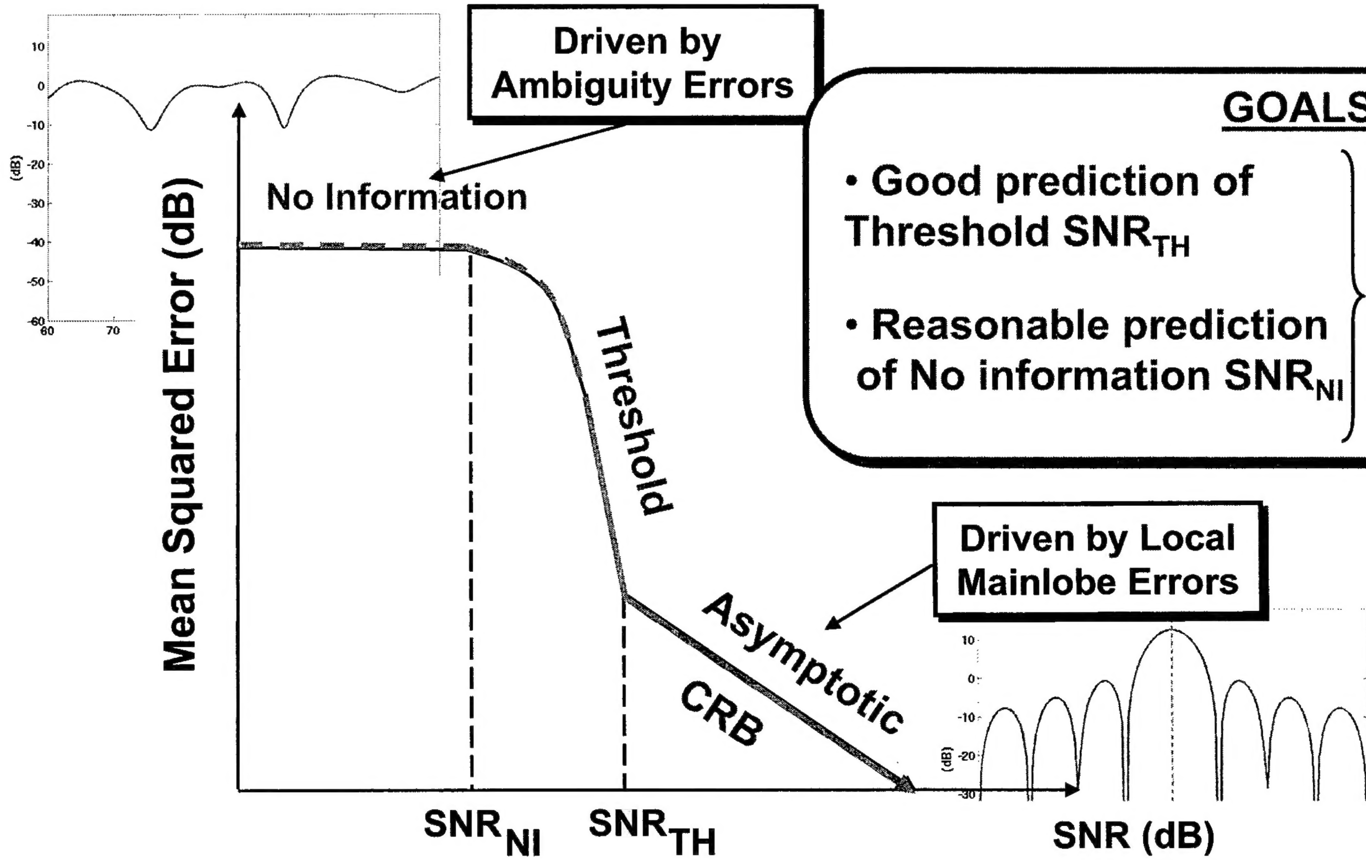
Outline

- Introduction
  - Problem
  - Previous Work
- Theory
- Numerical Results
- Conclusions

Goals of Analysis

- **Problem:**
  - Mean Squared Error (MSE) performance of Maximum-Likelihood (ML) signal parameter estimation unknown
    1. Colored Noise
    2. Finite Number of Colored Noise Only Training Samples
- **Goal:**
  - Develop robust theory for prediction of ML MSE
- **Proposed Method:**
  - Use Interval Error based method proposed by Van Trees 1968
    - Must derive/approximate probability of "interval error"

Typical Composite MSE Performance



GOALS:

- Good prediction of Threshold SNR_{TH}
- Reasonable prediction of No information SNR_{NI}

- Colored Noise
- Adaptive Training

- Three definitive regions of Signal-to-Noise-Ratio (SNR)
  - No Information, Threshold, and Asymptotic (CRB)
- Recall MSE = Estimator Variance + Estimator Bias

Previous Work

- K. Bell, Ph. D. George Mason University, 1995
- K. Bell, Y. Steinberg, Y. Ephraim, H. Van Trees, IEEE T-SP March 1997
- S. Pawlukiewic ... 1999
-
-
- F. Athley, IE...

What's New ?

- Colored Noise Allowed
- Colored Noise Only Finite training effects
- Exact two point error probabilities used

Ziv-Zakai Bounds

Method of Interval Error

- All previous work considered non-adaptive and white noise only case
- Error probabilities approximated via Chernoff Bounds

Outline

- Introduction
- Theory
  - Maximum-Likelihood Estimation (MLE)
  - Interval Error Based Method of MSE Prediction
- Numerical Results
- Conclusions

Maximum-Likelihood Signal Parameter Estimation

Data Model: $$\pi^{-N}\left|\mathbf{R}\right|^{-1}\exp\left\{-\left[\mathbf{x}-S\mathbf{v}(\theta)\right]^{H}\mathbf{R}^{-1}\left[\mathbf{x}-S\mathbf{v}(\theta)\right]\right\}$$

S unknown

ML Estimator: $$\theta_{ML}=\arg\max t_{MF}(\theta) \qquad t_{MF}(\theta)=\frac{\left|\mathbf{v}^{H}(\theta)\mathbf{R}^{-1}\mathbf{x}\right|^{2}}{\mathbf{v}^{H}(\theta)\mathbf{R}^{-1}\mathbf{v}(\theta)}$$

Clairvoyant Matched Filter

Data Model: $$\pi^{-N(L+1)}\left|\mathbf{R}\right|^{-(L+1)}\exp\left\{-\left[\mathbf{x}-S\mathbf{v}(\theta)\right]^{H}\mathbf{R}^{-1}\left[\mathbf{x}-S\mathbf{v}(\theta)\right]-\mathrm{tr}\left(\mathbf{R}^{-1}\mathbf{X}\mathbf{X}^{H}\right)\right\}$$

$\mathbf{R}$ unknown
S unknown

ML Estimator: $$\theta_{ML}=\arg\max t_{AMF}(\theta) \qquad t_{AMF}(\theta)=\frac{\left|\mathbf{v}^{H}(\theta)\hat{\mathbf{R}}^{-1}\mathbf{x}\right|^{2}}{\mathbf{v}^{H}(\theta)\hat{\mathbf{R}}^{-1}\mathbf{v}(\theta)} \qquad \hat{\mathbf{R}}\equiv\frac{1}{L}\mathbf{X}\mathbf{X}^{H}$$

Adaptive Matched Filter

- **Complex Gaussian data model: All snapshots N x 1**
  - **Arbitrary N x N Colored Covariance**
  - **Deterministic Signal ("Conditional")**
- **Colored noise only training samples available**
- **Single scalar signal parameter**
  - **Joint signal parameter estimation not considered**

Method of ML MSE Prediction: Based on Interval Errors

- **In general MSE can be written as the sum of two terms**

$$E\left\{\left(\hat{\theta}-\theta\right)^2\right\} = \Pr(\text{No Interval Error})E\left\{\left(\hat{\theta}-\theta\right)^2 \middle| \text{No Interval Error}\right\}$$
$$+\Pr(\text{Interval Error})E\left\{\left(\hat{\theta}-\theta\right)^2 \middle| \text{Interval Error}\right\}$$

- **MSE for Deterministic Signal Parameters**

SINR Loss!!!

$$\boxed{E\left\{\left(\hat{\theta}-\theta\right)^2 \middle| \theta_k\right\} \cong \left[1-\sum_{\substack{n=1\\n\neq k}}^{K} p\left(\hat{\theta}=\theta_n \middle| \theta_k\right)\right] \cdot \mathrm{CRB}(\theta_k) + \sum_{\substack{n=1\\n\neq k}}^{K} p\left(\hat{\theta}=\theta_n \middle| \theta_k\right)\left(\theta_n-\theta_k\right)^2}$$

- **Challenge is accurate calculation of error probabilities given by**

$$p\left(\hat{\theta}=\theta_n \middle| \theta_k\right) = ?$$

Union Bound (UB) Approximation: Interval Error Probabilities

- Recall the ML approach: $\hat{\theta} = \underset{\theta_1,\theta_2,\ldots,\theta_K}{\arg\max}\, t(\theta)$

- The probability of interval error is bounded by the relation

UNION BOUND

$$p\left(\hat{\theta} = \theta_n \middle| \theta_k\right) = \Pr\left\{ \bigcup_{\substack{k=1 \\ k \neq n}}^{K} \left[t(\theta_n) > t(\theta_k) \middle| \theta = \theta_k \right] \right\} \leq \sum_{\substack{k=1 \\ k \neq n}}^{K} \Pr\left[t(\theta_n) > t(\theta_k) \middle| \theta = \theta_k \right]$$

- UB is a useful tool for computation of error probabilities in Digital Communication Schemes
  - Approximation relies on <u>two point error probabilities</u>
- UB often over estimates error in “No Information” region of the MSE curve

Two Point Probabilities for the Matched Filter: R known

- Let array responses for two points be given by $\mathbf{V} = [\mathbf{v}(\theta_n), \mathbf{v}(\theta_k)]$
- Define the following matrices

$$\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{\mathbf{v}(\theta_n)\mathbf{R}^{-1}\mathbf{v}(\theta_n)}} & 0 \\ 0 & \frac{1}{\sqrt{\mathbf{v}(\theta_k)\mathbf{R}^{-1}\mathbf{v}(\theta_k)}} \end{bmatrix}$$

$$\mathbf{W} = \mathbf{R}^{-1}\mathbf{V}\mathbf{A} \qquad \mathbf{R}_{VX} \equiv \mathbf{A}\mathbf{V}^H\mathbf{R}^{-1}\mathbf{V}\mathbf{A}$$

$$\mathbf{R}_{VX}^{1/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{R}_{VX}^{1/2} = \mathbf{Q}_{VX}^H \Lambda_{VX} \mathbf{Q}_{VX}$$

- Defining the vector $\mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \mathbf{Q}_{VX}\mathbf{R}_{VX}^{-1/2}\mathbf{W}^H\mathbf{v}(\theta_k)$

The exact desired two point probabilities are given by

$$\boxed{\Pr\left[t_{MF}(\theta_n) > t_{MF}(\theta_k) \middle| \theta = \theta_k\right] = \Pr\left[\frac{\chi_1^2\left(|m_1|^2\right)}{\chi_1^2\left(|m_2|^2\right)} \leq \frac{-\lambda_{VX,2}}{\lambda_{VX,1}}\right]}$$

Expressible in terms of Marcum Q-function

Two Point Probabilities for the Adaptive Matched Filter: R unknown

- Let $\begin{bmatrix} t_{AMF}(\theta_n) \\ t_{AMF}(\theta_k) \end{bmatrix} = \begin{bmatrix} |y_{AMF,1}|^2 \\ |y_{AMF,2}|^2 \end{bmatrix}$; the desired probability can be written

$$\Pr\left[t_{AMF}(\theta_n) > t_{AMF}(\theta_k) \middle| \theta = \theta_k\right] = \Pr\left\{ \mathbf{y}_{AMF}^H \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}_{AMF} < 0 \right\}$$

- It can be shown that AMF outputs can be written equal in distribution

$$\mathbf{y}_{AMF} = \begin{bmatrix} y_{AMF,1} \\ y_{AMF,2} \end{bmatrix} \overset{d}{=} \begin{bmatrix} \sqrt{a_{11}} & \frac{a_{12}}{\sqrt{a_{11}}} \\ \frac{a_{21}}{\sqrt{a_{22}}} & \sqrt{a_{22}} \end{bmatrix} \left(\mathbf{V}^H \mathbf{R}^{-1} \mathbf{V}\right)^{-1/2} \mathbf{x}_{AMF} \quad \text{where}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \sim CW\left(L - N + 2, \mathbf{V}^H \mathbf{R}^{-1} \mathbf{V}\right) \text{ and } \mathbf{x}_{AMF} \sim CN_{2x1}\left(S \begin{bmatrix} \sqrt{\mathbf{v}(\theta_n) \mathbf{R}^{-1} \mathbf{v}(\theta_n)} \\ 0 \end{bmatrix}, \mathbf{I}_2 \cdot \frac{1}{\beta_{L-N+3,N-2}} \right)$$

- The necessary two point probabilities can be thus obtained

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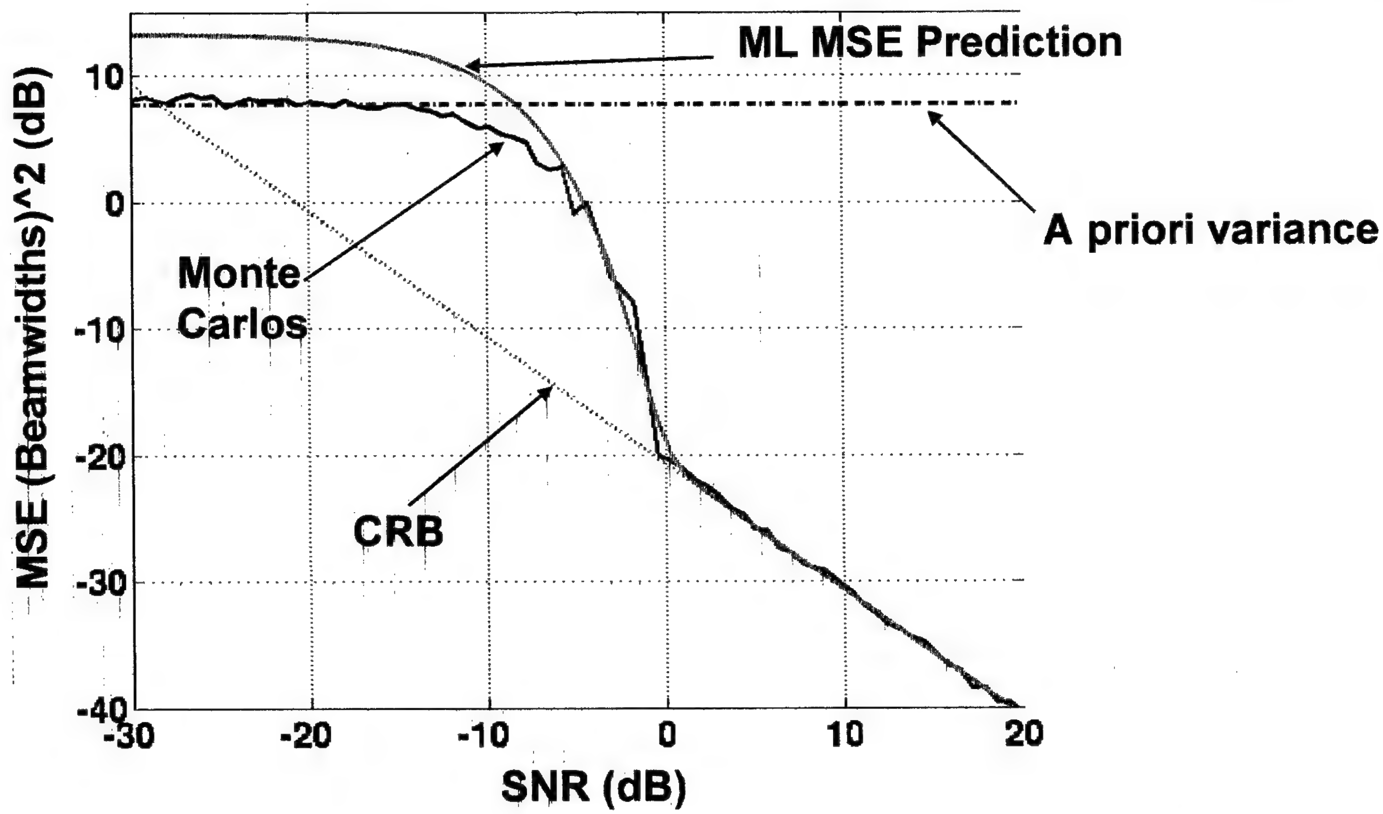
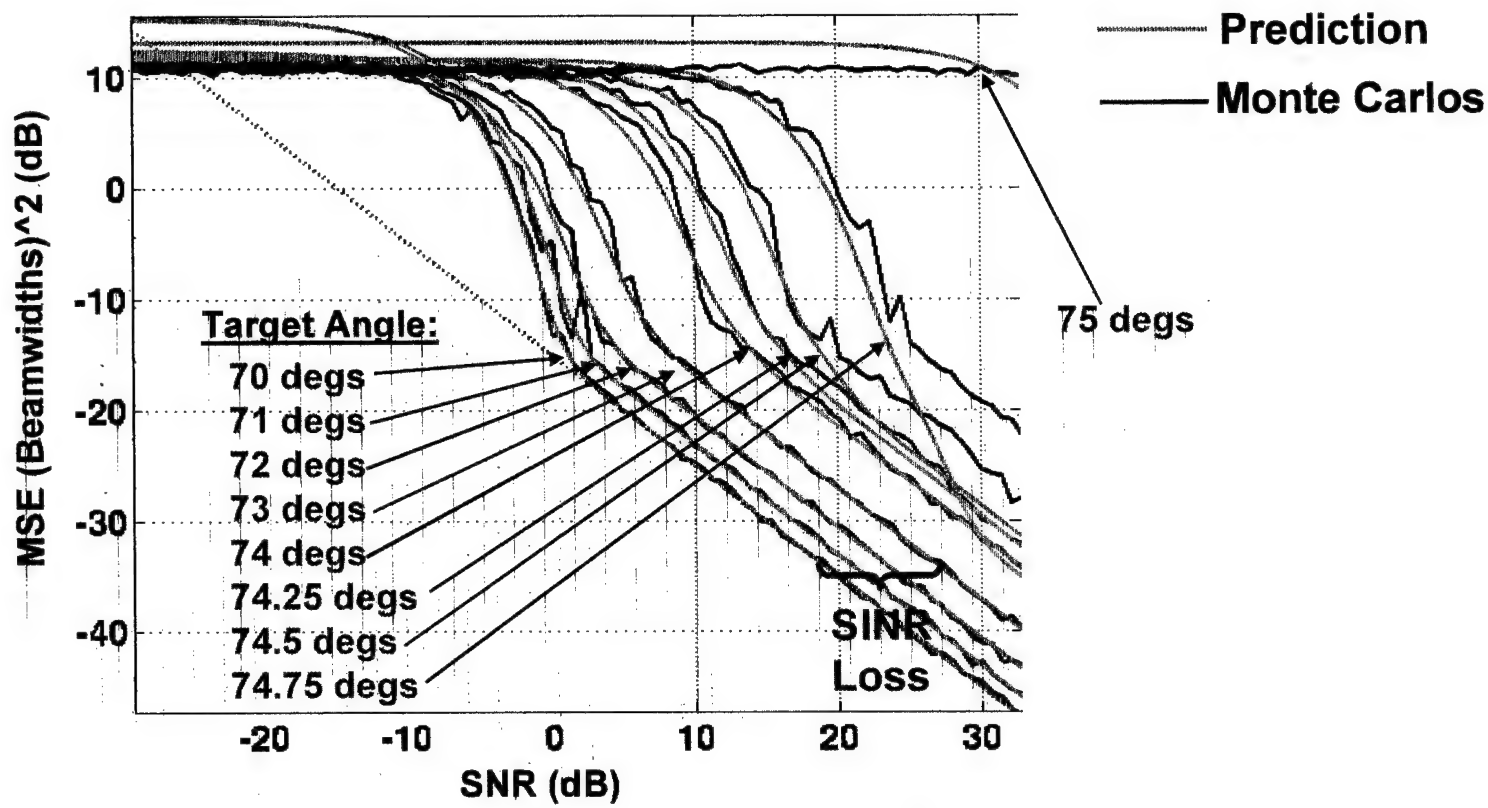
White Noise Example: R known



- N=18 element ULA, (λ/2.25) element spacing, broadside at 90 degs, endfire at 0 and 180 degs
- 0dB white noise

Colored Noise Example: R known



- N=18 element ULA, (λ/2.25) element spacing, broadside at 90 degs, endfire at 0 and 180 degs
- 0dB white noise plus 30dB Jammer at 75 degs

White Noise Example: R unknown

Need Figure

- $N=18$ element ULA, ($\lambda/2.25$) element spacing, broadside at 90 degs, endfire at 0 and 180 degs
- 0dB white noise
- Adaptive Training: $L = 1.5N$, $2N$, and $3N$

Colored Noise Example: R unknown

Need Figure

- N=18 element ULA, (λ/2.25) element spacing, broadside at 90 degs, endfire at 0 and 180 degs
- 0dB white noise plus 30dB Jammer at 75 degs
- Adaptive Training: L = 1.5N, 2N, and 3N

Conclusions

- Interval error method represents a viable and numerically efficient technique
  - Theory and simulation have very good match
  - UB overestimates MSE, however, in “No Information” region
- Two point probabilities have been computed in closed form
  - Colored Noise
  - Adaptive Finite Training Effects
- Established a the notion of SINR Loss for the parameter estimation problem

Future Work

- Explore tighter bounds on probability of interval errors than that given by the Union Bound
  - Expurgating terms of Union Bound, for example
- Extend to Stochastic / Unconditional signal models
- Generalize to vector signal parameters
- Comparisons with Bayesian Bound predictions
  - Ziv-Zakai, Weiss-Weinstein, etc.

Backups

Method of ML MSE Prediction: Based on Interval Errors

- **In general MSE can be written as the sum of two terms**

$$E\left\{\left(\hat{\theta}-\theta\right)^2\right\} = \Pr(\text{No Interval Error})\,E\left\{\left(\hat{\theta}-\theta\right)^2 \middle| \text{No Interval Error}\right\} + \Pr(\text{Interval Error})\,E\left\{\left(\hat{\theta}-\theta\right)^2 \middle| \text{Interval Error}\right\}$$

- **Deterministic Signal Parameters**

$$E\left\{\left(\hat{\theta}-\theta\right)^2 \middle| \theta_k\right\} = \Pr\left(\text{No Interval Error} \middle| \theta_k\right)\cdot \text{CRB}(\theta_k) + \sum_{\substack{n=1\\ n\neq k}}^{K} p\left(\hat{\theta}=\theta_n \,\middle|\, \theta_k\right)\left(\theta_n-\theta_k\right)^2$$

$$E\left\{\left(\hat{\theta}-\theta\right)^2\right\} = \int_{\hat{\Theta}} \left(\hat{\theta}-\theta\right)^2 p\left(\hat{\theta}\,|\,\theta\right) d\hat{\theta}$$

$$E\left\{\left(\hat{\theta}-\theta\right)^2 \middle| \theta\right\} = \int_{\hat{\Theta}:MAINLOBE} \left(\hat{\theta}-\theta\right)^2 p\left(\hat{\theta}\,|\,\theta\right) d\hat{\theta} + \int_{\hat{\Theta}:AMBIGUITIES} \left(\hat{\theta}-\theta\right)^2 p\left(\hat{\theta}\,|\,\theta\right) d\hat{\theta}$$

Two Point Probabilities for the Matched Filter: R known

- These probabilities are expressible in terms of the Marcum Q-function:

$$\Pr\left[t_{MF}(\theta_n) > t_{MF}(\theta_k) \mid \theta = \theta_k\right] = \Pr\left[\frac{\chi_1^2\left(|m_1|^2\right)}{\chi_1^2\left(|m_2|^2\right)} \le \frac{-\lambda_{VX,2}}{\lambda_{VX,1}}\right] =$$

$$\left[\frac{\lambda_{VX,2}}{\lambda_{VX,2} - \lambda_{VX,1}}\right] \cdot \left\{\frac{-\lambda_{VX,1}}{\lambda_{VX,2}} Q_1\left(\sqrt{\frac{2|m_1|^2 \lambda_{VX,1}}{\lambda_{VX,1} - \lambda_{VX,2}}}, \sqrt{\frac{2|m_2|^2 \lambda_{VX,2}}{\lambda_{VX,2} - \lambda_{VX,1}}}\right) + \left[1 - Q_1\left(\sqrt{\frac{2|m_2|^2 \lambda_{VX,2}}{\lambda_{VX,2} - \lambda_{VX,1}}}, \sqrt{\frac{2|m_1|^2 \lambda_{VX,1}}{\lambda_{VX,1} - \lambda_{VX,2}}}\right)\right]\right\}$$

Return